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Supersymmetry quantum mechanics and the asymptotic iteration method

**Damian Mikulski · Marcin Molski ·
Jerzy Konarski**

Abstract   A relation between the supersymmetry approach and the asymptotic iteration method is presented. A new form of the wave function is introduced. This approach has been successively employed to solve the Schrödinger equation for the Kratzer and Morse oscillators. The exact energy levels and the corresponding eigenfunctions have been calculated.

Keywords   Supersymmetry quantum mechanics · Asymptotic iteration method · Schrödinger equation · Kratzer oscillator · Morse oscillator

1 Introduction

Analytical solution of the Schrödinger equation for given potentials plays a very important role in quantum mechanics and theoretical spectroscopy. Unfortunately, only for a few potentials such as the Kratzer-Fues, Pöschl-Teller, Morse with zero angular momentum, Coulomb and harmonic oscillators this equation can be solved exactly. The Schrödinger equation can be solved using several alternative approaches. For example: the supersymmetry [1], the Nikiforov–Uvarov method [2], the Pekeris approximation [3], the shape invariance [4], the hypervirial perturbation method [5], the shifted and

D. Mikulski (✉) · M. Molski · J. Konarski
Department of Theoretical Chemistry, Faculty of Chemistry,
A. Mickiewicz University of Poznań, Poznań, Poland
e-mail: dmkwant@amu.edu.pl

M. Molski
e-mail: mamolski@amu.edu.pl

J. Konarski
e-mail: konarski@amu.edu.pl

modified shifted 1/N expansion methods [6], the variational method [7], the asymptotic iteration method (AIM) [8] and the perturbation ladder operators method [9].

Recently, it has been shown that the formalism of the (AIM) can be profitably employed to determine the exact analytical solutions of the Schrödinger equation and the corresponding eigenvalue. This method has been used to solve differential equations which are important for quantum mechanics. Successful application of this method have been made for many anharmonic oscillators. Barak et al. [10] presented the analytical solution of the radial Schrödinger equation for the Kratzer oscillator and the exponential cosine screened Coulomb potential [11]. Barakat et al. [12] applied this method to obtain the exact solutions of the one-dimensional Schrödinger equation with the Morse potential. For the numerical illustration these authors calculated the vibrational energies of the $^{7}Li_{2}$ molecule. The results are in agreement with the ones published in the literature. Al-Dossary [13] obtained an analytical eigenvalue for the rotating Morse oscillator by means of the asymptotic iteration method. For various diatomic molecules the energy eigenvalues have been calculated and results are compared with the supersymmetry, the Nikoforov-Uvarov, the modified shifted expansion 1/N method and the hypervirial perturbation method. The numerical energies calculated in this work are in good agreement with the numerical results obtained for the Morse potential with the centrifugal term. On the other hand Fernandez [14] using the AIM procedure obtained the solution of the Schrödinger equation with the sextic anharmonic oscillators. Recently, Boztosun and Karakoc [15] derived a general formula which simplifies the original asymptotic iteration method. Moreover, these authors showed a connection between AIM and Nikiforov-Uvarov method to solve the second order ordinary differential equations in the analytical way.

In the past decade the concept of supersymmetry quantum mechanics (SUSY QM) has been found to be very useful in analyzing the bound-state spectra. The key idea in SUSY QM consists in generating very useful relationships of the eigenvalue and eigenfunctions of the hierarchy of SUSY-partner Hamiltonian. Moreover, note that the Hamiltonian of the Schrödinger equation can be always factorized with help of the annihilation and creation operators. The factorization method was first introduced by Schrödinger [16] to solve the eigenvalue problem with the Coulomb potential algebraically. However, Infeld and Hull [17] generalized this method and successfully obtained a wide class of solvable potentials by considering different forms of the factorization. In this way a general review on the supersymmetry approach and the procedure of construction of a SUSY Hamiltonian hierarchy in order of a complete spectral resolution it is applied for various potential energy functions. The formalism of SUSY QM has been used by Fernandez et al. [18] for calculating the accurate energy eigenvalues of the Schrödinger equation. The SUSY Hamiltonian hierarchy was investigated by Sukumar [19] to solve the Schrödinger equation with the Pöschl-Teller potential. Various applications of SUSY QM in the theoretical spectroscopy were reported by Hamaker and Rau [20]. Also Bessis and Bessis [21] using concepts of the supersymmetry approach proposed a new algebraic procedure for the analytical solution of Schrödinger equation. Their formalism is based on the solution of the Riccati equation associated with a given wave function whose form is specific to each factorization type. Garnet Blado [22] using the fundamental concepts of the supersymmetric quantum mechanics has shown that the eigenvalues and radial eigenfunctions

for the ring-shaped Hartmann potential can be obtained exactly. On the other hand
Roy and Roychoudhury [23] derived the solutions of the Schrödinger equation with
the non polynomial oscillator within the supersymmetry approach

It is appropriate at this point to digress a bit and talk about the shape invariant
potential method. On the basis of the fundamental SUSY QM, Gendenshtein [4]
introduced this method to quantum mechanics. It has recently been shown that the
above-mentioned method has been profitably applied to many quantum mechanical
problems. With the help of shape solvable invariant potentials the Schrödinger equa-
tion can be solved analytically. Cooper et al. [24] using the shape invariant potentials
applied an operator transformation for the Pöschl-Teller potentials. In this works these
authors also found the Natanzon class of solvable potentials. Also Stahlhofen [25]
showed that the shape invariance condition and the factorization condition for Sturm-
Liouville eigenvalue problems are equivalent.

Recently, the formalism of SUSY has been extended for the semiclassical WKB
method [26]. This method is one of the most useful approximations of the energy
eigenvalues of the Schrödinger equation. It has a wider range of applicability than the
perturbation methods and 1/N expansion within SUSY QM [27]. Using the ideas of
SUSY with the lowest order WKB method Comtet et al. [28] have constructed the low-
est order SWKB quantization condition and proved that it yields energy eigenvalues
which are not only exact for large vibrational quantum numbers. Moreover, Inomata
and Junker [29] obtained the lowest order SWKB quantization condition in case SUSY
is broken. Recently, it has been shown [30,31] that for many classes of shape invariant
potentials and the Pöschl-Teller potential this lowest order SWKB calculation gives
the exact solvable spectrum.

The main purpose of the present work is to employ the supersymmetric quantum
mechanics to a new approach of the asymptotic iteration method. We show that our
approach can be successfully used to solve the Schrödinger equation for the Kratzer
and Morse oscillators. The work is organized as follows. In the second section we pres-
ent an improved version of the asymptotic iteration method which has been developed
by Boztosun and Karakoc [15]. In the third section the formalism of relation between
the supersymmetry method and the asymptotic iteration method is developed. In the
following section we consider the application of this approach to the Schrödinger equa-
tion for the rotating Kratzer and Morse oscillators. For this potentials we calculate the
energy eigenvalues and corresponding wave functions.

2 Basic concepts of the improvement asymptotic iteration method

Recently, Boztosun and Karakoc [15] derived a general formula which simplified
the original formalism of the asymptotic iteration method to find energy levels and
wave functions for the analytical solvable potentials [8]. Their concept is based on the
conversion of the homogenous second-order differential equation

$$\frac{d^2\Phi(r)_\upsilon}{dr^2} = \lambda_0(r)\frac{d\Phi(r)_\upsilon}{dr} + \alpha_0(r)\Phi(r)_\upsilon \tag{1}$$

to the equation of hypergeometric type. The functions $\lambda_0(r)$ and $\alpha_0(r)$, are sufficiently differentiable. For given exactly solvable potential energy functions the Schrödinger equation can be converted into this form. In this sense the Eq. 1 after employing the following forms for $\lambda_0(r)$ and $\alpha_0(r)$ [15]

$$\lambda_0(r) = -\frac{f(r)}{\sigma(r)}, \quad \alpha_0(r) = -\frac{\varepsilon_n}{\sigma(r)} \tag{2}$$

can be transformed to the form [15]

$$\frac{d^2\Phi(r)_\upsilon}{dr^2} = -\frac{f(r)}{\sigma(r)}\frac{d\Phi(r)_\upsilon}{dr} - \frac{\varepsilon_n}{\sigma(r)}\Phi(r)_\upsilon. \tag{3}$$

Note that $\sigma(r)$ is polynomial at most of second degree and f(r) is a first degree at most polynomial. However, ε_υ is a constant which includes the energy eigenvalue. Adopting to this approach the quantization conditions of the standard asymptotic iteration method [8]

$$\delta_k(r) = \lambda_k(r)\alpha_{k-1}(r) - \lambda_{k-1}(r)\alpha_k(r), \quad k = 1, 2, 3, \ldots \tag{4}$$

the energy eigenvalues can be determined using the following relationship [15]

$$\varepsilon_\upsilon = -\upsilon\frac{df(r)}{dr} - \frac{\upsilon(\upsilon-1)}{2}\frac{d^2\sigma(r)}{dr^2} \tag{5}$$

in which $\upsilon = 0, 1, 2, \ldots$ and denotes the vibrational quantum number. The energy eigenvalue are calculated from this equation if the problem is exactly solvable.

Accordingly to the original AIM procedure the functions $\lambda_k(r)$ and $\alpha_k(r)$ in Eq. 4 are calculated by the following recurrence relations [8]

$$\lambda_k(r) = \frac{d\lambda_{k-1}(r)}{dx} + \alpha_{k-1}(r) + \lambda_0(r)\lambda_{k-1}(r) \tag{6}$$

$$\alpha_k(r) = \frac{d\alpha_{k-1}(r)}{dr} + \alpha_0(r)\lambda_{k-1}(r) \tag{7}$$

This method can be successively applied to generate the wave function using the following wave function generator [15]

$$\Phi(r)_\upsilon = \frac{1}{\rho(r)}\frac{d^\upsilon}{dr^\upsilon}\left[\sigma^\upsilon(r)\rho(r)\right] \tag{8}$$

in which $\rho(r)$ is a solution of the following first-degree differential equation [15]

$$\frac{d}{dr}\left[\sigma(r)\rho(r)\right] = f(r)\rho(r) \tag{9}$$

3 The supersymmetry and asymptotic iteration method

In this chapter we turn our attention to the relation between the method described method in the previous section with supersymmetry method in quantum mechanics (SUSYQM). The crucial point in the approach proposed is the assumption that the wave function can be specified in a new form

$$\Psi(r)_\upsilon = \exp\left[\int W_0(r)dr + \beta(r)\right]\Phi(r)_\upsilon \tag{10}$$

in which $W_0(r)$ in SUSYQM [26] is interpreted as Witten superpotential [32] which permits construction of the supersymmetric Schrödinger equation straightforward to analytical solutions. $\Phi(r)_\upsilon$ is a new function to be determined with the property $\Phi(r)_0 = 1$. We assume the function $\beta(r)$ in the form ensuring that the wave function leads to correct asymptotic value which may have physical interpretation. It should be emphasized that assuming the wave function in the form of $\Psi_n(x) = \exp\left(\int W_0(x)dx\right)g_n(x) = \Psi_0(x)g_n(x)$ [34] is incorrect as for any vibrational state is contains the energy of the ground state. Consequently, it does not have physical interpretation. The expression for the wave function derived in this work is devoid of this drawback, which will be explained in the following sections of the paper.

It is easy to demonstrate that the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r)\right]\Psi(r)_\upsilon = E_\upsilon\Psi(r)_\upsilon \tag{11}$$

under the following transformation

$$W_\upsilon(r) = \frac{d}{dr}\ln\Psi(r)_\upsilon \tag{12}$$

can be transformed into the well-known Riccati equation

$$-\frac{\hbar^2}{2m}\left[W_\upsilon^2(r) + \frac{dW_\upsilon(r)}{dr}\right] + V(r) = E_\upsilon \tag{13}$$

in which V(r) is a given exactly solvable potential. Note that the above equations are valid for every υ, which means that in particular

$$-\frac{\hbar^2}{2m}\left[W_0^2(r) + \frac{dW_0(r)}{dr}\right] + V(r) = E_0 \tag{14}$$

remains valid for $\upsilon = 0$. The relationship (14) plays an important role because it permits determination of the superpotential and the corresponding ground state energy for a given exactly solvable oscillators.

In order to demonstrate how this method works we introduce in our study the following structure for the superpotential

$$W_0(r) = \gamma + \eta g(r) \qquad (15)$$

It is clear that the forms of $g(r)$ and $W_0(r)$ depend on the representation of the oscillator potent, hence explicit expressions for this functions will be determined for a given form of the potential energy function. Having the form of superpotential (15) we substitute it for the Riccati differential equation. After simple calculation we obtain constants γ and η. In the next step we define function $\beta(r)$ and substitute these function with the superpotential (15) into the wave function (10). If we now insert this function into the Schrödinger equation (11) we transform this equation to the following second order differential equation

$$-\frac{\hbar^2}{2m}\left\{ \frac{d^2\Phi(r)_\upsilon}{dr^2} + 2\left(W_0(r) + \frac{d\beta(r)}{dr}\right)\frac{d\Phi(r)_\upsilon}{dr} + \left[\frac{dW_0(r)}{dr} + \frac{d^2\beta(r)}{dr^2}\right.\right.$$

$$\left.\left. + \left(W_0(r) + \frac{d\beta(r)}{dr}\right)^2 \right]\Phi(r)_\upsilon \right\} + [V(r) - E_\upsilon]\,\Phi(r)_\upsilon = 0 \qquad (16)$$

For a few exactly solvable potentials this equation can be transformed to the hypergeometric type. Moreover, if we compare this equation with Eq. 3 we obtain the functions f(r), $\sigma(r)$ and ε_υ. Knowing the properties of this functions we can generate the energy levels and the corresponding eigenfunctions using Eqs. 5 and 8.

4 The Schrödinger equation with the rotating Kratzer oscillator

In this chapter we apply our approach to solve the radial Schrödinger equation with the rotating Kratzer oscillator

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - 2D_e\left(\frac{r_e}{r} - \frac{r_e^2}{2r^2}\right) + \frac{J(J+1)\hbar^2}{2mr^2}\right]\Psi(r)_{\upsilon J} = E_{\upsilon J}\Psi(r)_{\upsilon J} \quad (17)$$

in which D_e is the dissociation energy of the molecule, r_e is the equilibrium internuclear separation, r denotes the internuclear separation and J = 0, 1, 2,… is the rotational quantum number. The starting point to realize this aim is the assumption of the wave function in the general form

$$\Psi(r)_{\upsilon J} = \exp\left[\int W_0(r)dr + \beta(r)\right]\Phi(r)_{\upsilon J} \qquad (18)$$

in which the superpotential $W_0(r)$ can be specified in the following formula

$$W_0(r) = \gamma + \frac{\eta}{r} \qquad (19)$$

In this way Eq. 14 with this superpotential may be written

$$-\frac{\hbar^2}{2m}\left(\gamma^2 + \frac{\eta^2 - \eta}{r^2} + \frac{2\gamma\eta}{r}\right) - E_{0J} = 2D_e\left(\frac{r_e}{r} - \frac{r_e^2}{2r^2}\right) - \frac{J(J+1)\hbar^2}{2mr^2}. \quad (20)$$

The simple calculation reveals that

$$\eta = \Omega + 1 \ \vee \ \eta = \Omega' + 1 \text{ and } \gamma = \left(-\frac{E_{0J}2m}{\hbar^2}\right)^{1/2} \vee \ \gamma = -\left(-\frac{E_{0J}2m}{\hbar^2}\right)^{1/2} \quad (21)$$

in which

$$\Omega = -\frac{1}{2} + \frac{\sqrt{1 + \frac{8m}{\hbar^2}\left[D_e r_e^2 + \frac{\hbar^2 J(J+1)}{2m}\right]}}{2}, \ \Omega' = -\frac{1}{2} - \frac{\sqrt{1 + \frac{8m}{\hbar^2}\left[D_e r_e^2 + \frac{\hbar^2 J(J+1)}{2m}\right]}}{2} \quad (22)$$

It is readily seen that the wave function vanished at r = 0, corresponding to a strong repulsion between the two atoms. Moreover, at r = ∞ normalizable solutions in bound state behave as $e^{-\gamma_{\upsilon J}r}$. Therefore, it is reasonable to set

$$\beta(r) = -\gamma r - \gamma_{\upsilon J}r, \quad \gamma = \left(-\frac{E_{0J}2m}{\hbar^2}\right)^{1/2}, \quad \eta = \Omega + 1 \quad (23)$$

where $\gamma_{\upsilon J} = \left(-E_{\upsilon J}2m/\hbar^2\right)^{1/2}$. Putting Eqs. 19 and 23 into 18 leads to

$$\Psi(r)_{\upsilon J} = r^{\Omega+1}\exp\left(-\gamma_{\upsilon J}r\right)\Phi(r)_{\upsilon J} \quad (24)$$

If we insert this asymptotic wave function into Eq. 17 we have the second order homogenous linear differential equations in the form

$$\frac{d^2\Phi(r)_{\upsilon J}}{dr^2} + 2\left(-\gamma_{\upsilon J} + \frac{\Omega+1}{r}\right)\frac{d\Phi(r)_{\upsilon J}}{dr} + \left[\frac{4D_e r_e m}{\hbar^2 r} - \frac{2\gamma_{\upsilon J}(\Omega+1)}{r}\right.$$
$$\left. + \frac{(\Omega+1)^2 - (\Omega+1) - \frac{2m}{\hbar^2}\left(D_e r_e^2 + \frac{\hbar^2 J(J+1)}{2m}\right)}{r^2}\right]\Phi(r)_{\upsilon J} \quad (25)$$

Making some algebraic manipulations, we reduce the Eq. 25 to the equation given bellow which is the general type of the Kummer's equation of the confluent hyper-

geometric function

$$\frac{d^2\Phi(r)_{\upsilon J}}{dr^2} = 2\left(\gamma_{\upsilon J} - \frac{\Omega+1}{r}\right)\frac{d\Phi(r)_{\upsilon J}}{dr} + 2\left[\frac{\kappa}{r} + \frac{\gamma_{\upsilon J}(\Omega+1)}{r}\right]\Phi(r)_{\upsilon J}$$

(26)

where

$$\kappa = \frac{-D_e r_e 2m}{\hbar^2}$$

(27)

Comparing this result with Eq. 3 we realize that

$$f(r) = -2\left[\gamma_{\upsilon J}r - (\Omega+1)\right], \quad \sigma(r) = r, \quad \varepsilon_\upsilon = -2\left[\kappa + \gamma_{\upsilon J}(\Omega+1)\right] \quad (28)$$

Introducing Eq. 28 into 5 one gets the relation

$$-2\left[\kappa + \gamma_{\upsilon J}(\Omega+1)\right] = \upsilon\frac{d}{dr}\left[2\gamma_{\upsilon J}r - 2(\Omega+1)\right]$$

(29)

which leads to the energy levels. After some calculations the corresponding expressions of the energy eigenvalues turn out be

$$E_{\upsilon J} = -\frac{2m D_e^2 r_e^2}{\hbar^2}\frac{1}{(\upsilon+\Omega+1)^2}$$

(30)

As it can be seen that this representation of the energy eigenvalue is identical with the findings in the work [10].

By means of the concept introduced in preceding calculations we are now ready to construct the radial eigenfunction for the Kratzer oscillator. Taking into account Eq. 8 we obtain the following forms of the first iterative functions:

$$\Phi(r)_{0J} = 1 \tag{31}$$

$$\Phi(r)_{1J} = f(r) \tag{32}$$

$$\Phi(r)_{2J} = f^2(r) + f(r)\sigma'(r) + f'(r)\sigma(r) + \sigma(r)\sigma''(r) \tag{33}$$

$$\Phi(r)_{3J} = f^3(r) + 3f^2(r)\sigma'(r) + 2f(r)\left[\sigma''(r)\right]^2 + 3f(r)f'(r)\sigma(r)$$
$$+ 4f'(r)\sigma(r)\sigma'(r) + 5f(r)\sigma(r)\sigma''(r) + 6\sigma\sigma'\sigma'' \tag{34}$$

where the sign prim stands for differentiation over the variable r. It is clear that when the above functions are generalized, $\Phi(r)_{\upsilon J}$ can be specified as follows

$$\Phi(r)_{\upsilon J} = C_{\upsilon J} F\left[\Omega + 1 - \frac{a^2}{b}, 2(\Omega+1); \frac{2br}{r_e}\right]$$

(35)

where $a^2 = 2mr_e^2 D_e/\hbar^2$, $b = \sqrt{-\frac{2mr_e^2 E_{vJ}}{\hbar^2}}$, $C_{vJ} = (\Omega + v + 1)^v \left[\prod_{l=0}^{v-1} (2v + 2 + l) \right]$ and $F(c, d; x)$ stands for confluent hypergeometric (or Kummer) series defined by the Gamma function

$$F(c, d; x) = \sum_{n=0}^{\infty} \frac{\Gamma(c+n)\Gamma(d)x^n}{\Gamma(d+n)\Gamma(c)n!} \tag{36}$$

It is obvious that the regularity of the wave function at r = 0 implies that the series (36) must be a polynomial. This fact leads to

$$\Omega + 1 - \frac{a^2}{b} = -v \tag{37}$$

Consequently, the solution in the arbitrary normalization of the Schrödinger equation (17) can be written as

$$\Psi(r)_{vJ} = C_{vJ} r^{\Omega+1} \exp\left(-\gamma_{vJ} r\right) F\left[-v, 2(\Omega + 1); \frac{2br}{r_e}\right] \tag{38}$$

It may be seen than that this result is also in excellent agreement with the findings achieved with the use the standard version of the asymptotic iteration method [10].

5 The Schrödinger equation with the Morse oscillator

In this section we turn our attention to the solution of the Schrödinger equation for the Morse oscillator

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + D_e \left(e^{-2\alpha x} - 2e^{-\alpha x}\right) \right] \Psi(x)_v = E_v \Psi(x)_v \tag{39}$$

in which α is the range factor, D_e is the dissociation energy and $x = r - r_e$ whereas the parameter r_e denotes the equilibrium internuclear separation. After changing the variable [33]

$$\xi = \frac{2\sqrt{2mD_e}}{\alpha \hbar} e^{-\alpha x} \tag{40}$$

we can rewrite the Schrödinger equation (39) in the following form [33]

$$\frac{d^2\Psi(\xi)_v}{d\xi^2} + \frac{1}{\xi}\frac{d\Psi(\xi)_v}{d\xi} + \left(-\frac{1}{4} + \frac{v + \lambda_v + \frac{1}{2}}{\xi} - \frac{\lambda_v^2}{\xi^2}\right)\Psi(\xi)_v = 0 \tag{41}$$

where

$$\lambda_\upsilon = \frac{\sqrt{-2mE_\upsilon}}{\alpha\hbar}, \quad \upsilon = \frac{\sqrt{2mD_e}}{\alpha\hbar} - \lambda_\upsilon - \frac{1}{2} \tag{42}$$

Equation 41 can be easily transformed to the following canonical form

$$\frac{d^2\varphi(\xi)_\upsilon}{d\xi^2} + \left(-\frac{1}{4} + \frac{\upsilon + \lambda_\upsilon + \frac{1}{2}}{\xi} - \frac{\lambda_\upsilon^2}{\xi^2} + \frac{1}{4\xi^2}\right)\varphi(\xi)_\upsilon = 0 \tag{43}$$

in which $\varphi(\xi)_\upsilon$ satisfies the following transformation by Liouville

$$\Psi(\xi)_\upsilon = \varphi(\xi)_\upsilon \exp\left[-\frac{1}{2}\int \frac{1}{\xi}d\xi\right] \tag{44}$$

Let us now construct the wave function. This can be accomplished if we choose

$$\Psi(\xi)_\upsilon = \xi^{-1/2} \exp\left[\int W_0(\xi)d\xi + \beta(\xi)\right]\Phi(\xi)_\upsilon \tag{45}$$

in which $\beta(\xi)$ and $\Phi(\xi)_\upsilon$ are unknown functions which we determine in the next part of our study. Using the following transformation

$$W_0(\xi) = \frac{d}{d\xi}\ln\varphi(\xi)_0 \tag{46}$$

one gets the Riccati equation

$$\left[W_0^2(\xi) + \frac{dW_0(\xi)}{d\xi}\right] = \frac{1}{4} - \frac{\lambda_0 + \frac{1}{2}}{\xi} + \frac{\lambda_0^2}{\xi^2} - \frac{1}{4\xi^2} \tag{47}$$

where $\lambda_0 = \sqrt{-2mE_0}/\alpha\hbar$.

In order to solve the Eq. 47, we assume the superpotential $W_0(\xi)$ in the form

$$W_0(\xi) = \gamma + \frac{\eta}{\xi} \tag{48}$$

Introducing Eq. 48 into 47 we obtain the following equation

$$\left[\gamma^2 + \frac{2\eta\gamma}{\xi} + \frac{\eta^2 - \eta}{\xi^2}\right] = \frac{1}{4} - \frac{\lambda_0 + \frac{1}{2}}{\xi} + \frac{\lambda_0^2}{\xi^2} - \frac{1}{4\xi^2} \tag{49}$$

Equating coefficients in Eq. 49 we arrive at the following relations

$$\gamma = \frac{1}{2} \vee \gamma = -\frac{1}{2} \quad \text{and} \quad \eta = \frac{1 - \sqrt{1 + 4\left(\lambda_0^2 - \frac{1}{4}\right)}}{2} \vee \eta = \frac{1 + \sqrt{1 + 4\left(\lambda_0^2 - \frac{1}{4}\right)}}{2} \tag{50}$$

Note that the differential equation (41) has singularities at $\xi = 0$ and $\xi = \infty$. In the vicinity of $\xi = 0$ the wave function (44) to be proportional to ξ^{λ_v} and at infinity this wave function is proportional to $e^{-1/2\xi}$. This behavior of the solution of Eq. 41 suggests the following relationships

$$\beta(\xi) = -\xi + \ln \xi^{-\eta + \lambda_v + 1/2}, \quad \gamma = \frac{1}{2} \tag{51}$$

By the substitution of this function and the superpotential (48) into Eq. 45 we find the asymptotic wave function in the form

$$\Psi(\xi)_v = \xi^{\lambda_v} \exp\left(-\frac{\xi}{2}\right) \Phi(\xi)_v \tag{52}$$

Therefore, if we substitute the wave function (52) into Eq. 39 this equation can be transformed in the following hypergeometric equation [33]

$$\xi \frac{d^2 \Phi(\xi)_v}{d\xi^2} + (2\lambda_v + 1 - \xi) \frac{d\Phi(\xi)_v}{d\xi} + v\Phi(\xi)_v = 0 \tag{53}$$

Thus, we see that this equation can be solved using proposed method. By applying the method described in the previous chapter, which generates the energy eigenvalue, we obtain the following expression for the functions $f(\xi)$, $\sigma(\xi)$ and ε_v

$$f(\xi) = 2\lambda_v + 1 - \xi, \ \sigma(\xi) = \xi, \ \varepsilon_v = v \tag{54}$$

Introduction of this functions into Eq. 5 leads to a relation from which we determine the energy eigenvalue

$$\frac{\sqrt{2mD_e}}{\alpha\hbar} - \frac{\sqrt{-2mE_v}}{\alpha\hbar} - \frac{1}{2} = -v\frac{d}{d\xi}(2\lambda_v + 1 - \xi) \tag{55}$$

Finally, after elementary calculations we arrive at the energy formula

$$E_v = -D_e\left[1 - \frac{\alpha\hbar}{\sqrt{2mD_e}}\left(v + \frac{1}{2}\right)\right]^2 \tag{56}$$

It should be pointed that the expression (56) is fully equivalent to those obtained previously in the classical work by Landau and Lifszyc [33].

Proceeding along the lines of the method proposed in the previous sections the function $\Phi(\xi)_v$ can be specified of the Kummer's confluent hypergeometric function

$$\Phi(\xi)_v = (-1)^v \frac{\Gamma\left(\frac{2\delta_v}{\alpha} + v + 1\right)}{\Gamma\left(\frac{2\delta_v}{\alpha} + 1\right)} F(-v, 2\lambda_v + 1, \xi) \tag{57}$$

in which $\delta_\upsilon = \sqrt{\frac{2mD_e}{\hbar^2}} - \alpha \left(\upsilon + \frac{1}{2}\right)$.

Hence the solution in the arbitrary normalization for the Morse oscillator can be expressed in the well-known form

$$\Psi(\xi)_\upsilon = (-1)^\upsilon \frac{\Gamma\left(\frac{2\delta_\upsilon}{\alpha} + \upsilon + 1\right)}{\Gamma\left(\frac{2\delta_\upsilon}{\alpha} + 1\right)} \xi^{\lambda_\upsilon} \exp\left(-\frac{\xi}{2}\right) F(-\upsilon, 2\lambda_\upsilon + 1, \xi) \qquad (58)$$

6 Conclusions

In this work the relation between the supersymmetry approach and the new version of the asymptotic iteration method has been developed. This approach can be applied to generate the solution of the Schrödinger equation. To illustrate the usefulness of the approach proposed we have found the explicit solution of the Schrödinger equation for the rotating Kratzer and Morse oscillators. In our study we have also constructed the Witten superpotential for these oscillators by means of the theory of the differential Riccati equation. Moreover we have introduced a new general form of the wave function. We think that this approach can be used to find the solutions of new exactly solvable oscillators which are widely applied in spectroscopy of rotation-vibrational diatomic systems.

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